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ONE CLASS OF PARTIALLY INVARIANT SOLUTIONS OF THE NAVIER–STOKES EQUATIONS

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A family of partially invariant solutions of the Navier-Stokes equations of rank 2 and defect 2 is considered. These solutions describe the three-dimensional unsteady motions of a viscous incompressible fluid in which the vertical velocity component and the pressure are independent of the horizontal coordinates. In particular, they can be interpreted as flows in a horizontal layer, one boundary of which is the free surface.

1. Invariant and Partially Invariant Solutions of the Navier–Stokes Equations. We analyze the system of Navier–Stokes equations

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \Delta \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{1.1}$$

which is considered from the group-theoretic standpoint. Here $\boldsymbol{u} = (u, v, w)$ is the velocity vector, p is the pressure, t is the time, and ∇ and Δ are the gradient and the Laplacian in variables $\boldsymbol{x} = (x, y, z)$, respectively.

System (1.1) admits the infinite-dimensional group of transformations [1] that produces its numerous invariant solutions (see [2] and references therein). Many of these solutions have been known for a long time; however, their systematic analysis has become possible due to the development of up-to-date methods of group analysis of differential equations [3]. At the same time, as far back as the beginning of the twentieth century, exact solutions of system (1.1) were found, the group nature of which remained unclear until the 1970s. The most known of them is the von Kármán solution [4]. It was found that the von Kármán solution and its analogs, though having also a group origin, are not the invariant solutions [5]: they relate to the class of partially invariant solutions [6] of the Navier–Stokes system.

A systematic analysis of the partially invariant solutions of system (1.1) has not been performed so far, although some representatives of this class were studied in detail [7–11]. As is well known, the main difficulty in investigating the partially invariant solutions is the analysis of compatibility of the arising overdeterminate system of equations. In the studies mentioned above, this analysis did not require special efforts. In the present paper, the class of partially invariant solutions of system (1.1) of rank 2 and defect 2 is studied, where the procedure of reducing an overdeterminate system of equations for four independent variables to a passive system is not trivial.

2. New Class of Partially Invariant Solutions of System (1.1). Let us consider the group G^4 of transformations of Euclidean space $\Re^8$ with the coordinates x, y, z, t, u, v, w, and p generated by the operators $X = \partial_x$, $Y = \partial_y$, $U = t\partial_x + \partial_u$, and $V = t\partial_y + \partial_v$. Obviously, the group G^4 is admitted by the Navier–Stokes system; however, none of the invariant solutions of this system corresponds to it. Indeed, the complete set of invariants of the group G^4 involves z, t, w, and p, so that the rank of the relative Jacobi matrix [3] is 2, and the number of invariants which do not involve unknown functions is also equal to 2. This

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allows one to find a regular, partially invariant solution [12] of rank 2 and defect 2 of system (1.1) relative to the group G^4 in the form

$$w = 2f(z,t), \quad p = p(z,t), \quad u = u(x,y,z,t), \quad v = v(x,y,z,t). \tag{2.1}$$

This class of solutions for the gas dynamic equations is considered in [13]. Instead of the velocity components u and v, it is more convenient to choose the functions $\widehat{u}(x,y,z,t)$ and $\widehat{v}(x,y,z,t)$, which are related to u and v by the formulas

$$u = \widehat{u} - x\frac{\partial f}{\partial z}, \quad v = \widehat{v} - y\frac{\partial f}{\partial z}.$$

The second equation of system (1.1) with (2.1) is then rewritten in the form

$$\frac{\partial \widehat{u}}{\partial x} + \frac{\partial \widehat{v}}{\partial y} = 0,$$

and it can be satisfied by introducing the analog of the stream function $\psi = \psi(x,y,z,t)$ by the formulas

$$\widehat{u} = \frac{\partial \psi}{\partial y}, \quad \widehat{v} = -\frac{\partial \psi}{\partial x}.$$

Moreover, the first two scalar equations equivalent to the vector equation of system (1.1) assume the form

$$\begin{gathered}\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} + 2f\psi_{yz} - x(f_{zt} + f_z\psi_{xy} + 2ff_{zz} - f_z^2) - yf_z\psi_{yy} = \Delta\psi_y - xf_{zzz}, \\ -\psi_{xt} - \psi_y\psi_{xx} + \psi_x\psi_{xy} - 2f\psi_{xz} - y(f_{zt} - f_z\psi_{xy} + 2ff_{zz} - f_z^2) + xf_z\psi_{xx} = -\Delta\psi_x - yf_{zzz},\end{gathered} \tag{2.2}$$

while the third equation is separated and serves to determine the pressure $p(z,t)$ if the function $f(z,t)$ is known:

$$p_z + 2f_t - 2f_{zz} + 4ff_z = 0.$$

We note that summation of the first equation (2.2) differentiated with respect to x and the second equation differentiated with respect to y gives the equation

$$\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 - \frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2 f}{\partial z\partial t} + 2f\frac{\partial^2 f}{\partial z^2} - \frac{\partial^3 f}{\partial z^3} - \left(\frac{\partial f}{\partial z}\right)^2. \tag{2.3}$$

In this equation, the right-hand side depends only on z and t, and, therefore, it can be considered as the Monge–Ampere equation with a constant (depending on z and t) right-hand side.

In this paper, we consider the hyperbolic case where the right-hand side in the Monge–Ampere equation (2.3) is nonnegative. We denote it by

$$\alpha^2(z,t) \equiv \frac{\partial^2 f}{\partial z\partial t} + 2f\frac{\partial^2 f}{\partial z^2} - \frac{\partial^3 f}{\partial z^3} - \left(\frac{\partial f}{\partial z}\right)^2.$$

It is well known [14] that, in this case, the Monge–Ampere equation has the first integral

$$\frac{\partial g}{\partial y} = 2\alpha x + G\Big(z,t,\frac{\partial g}{\partial x}\Big), \tag{2.4}$$

where $g(x,y,z,t) = \psi(x,y,z,t) + xy\alpha(z,t)$, and $G(z,t,\xi)$ is an arbitrary function of integration. Substituting this representation into the first equation of (2.2) and combining with the second equation, we obtain the equality

$$S \equiv b_4 g_{xx}^2 + b_5 g_{xz}^2 + b_1 g_{xx} + b_2 g_{xz} - b_3 = 0, \tag{2.5}$$

where

$$b_1 = 4\alpha\frac{\partial^2 G}{\partial \xi^2}\frac{\partial G}{\partial \xi}, \quad b_2 = 2\frac{\partial^2 G}{\partial z\partial \xi}, \quad b_4 = \frac{\partial^2 G}{\partial \xi^2}\left(\left(\frac{\partial G}{\partial \xi}\right)^2 + 1\right), \quad b_5 = \frac{\partial^2 G}{\partial \xi^2},$$

$$b_3 = \widehat{\alpha}\Big(x + y\frac{\partial G}{\partial \xi}\Big) + (f_z - \alpha)\Big(\xi\frac{\partial G}{\partial \xi} - G\Big) + \frac{\partial G}{\partial t} + 2f\frac{\partial G}{\partial z} - \frac{\partial^2 G}{\partial z^2} - 4\alpha^2\frac{\partial^2 G}{\partial \xi^2},$$

$$\widehat{\alpha} \equiv \frac{\partial \alpha}{\partial t} + 2f\frac{\partial \alpha}{\partial z} - \frac{\partial^2 \alpha}{\partial z^2} - 2\alpha\frac{\partial f}{\partial z}.$$

Furthermore, differentiating Eq. (2.5) with respect to x and y and making the combination $D_yS - G_\xi D_xS - 2g_{xx}G_{\xi\xi}S = 0$, we obtain the equality containing the quadratic polynomial in g_{xx}, and g_{xz}:

$$\alpha\frac{\partial^3 G}{\partial \xi^3}\Big(\Big(\frac{\partial G}{\partial \xi}\Big)^2 + 1\Big)g_{xx}^2 + \alpha\frac{\partial^3 G}{\partial \xi^3}g_{xz}^2 + c_1 g_{xx} + c_2 g_{xz} + c_3 = 0. \tag{2.6}$$

Here

$$c_1 = \widehat{\alpha}\frac{\partial^2 G}{\partial \xi^2}\Big(x + y\frac{\partial G}{\partial \xi}\Big) + \widehat{c}_1, \quad c_2 = 2\Big(\alpha_z\frac{\partial^2 G}{\partial \xi^2} + \alpha\frac{\partial^3 G}{\partial z \partial \xi^2}\Big), \quad c_3 = -y\alpha\widehat{\alpha}\frac{\partial^2 G}{\partial \xi^2} + \widehat{c}_3$$

with certain functions $\widehat{c}_i$ ($i = 1, 3$) which do not depend on x and y explicitly. We do not give some of the functions because of their cumbersome form. Hereafter, to treat the involved mathematical expressions, we use the system of analytical calculations REDUCE [15]. For further investigation, it is necessary to consider different variants which depend on the value of the function $\partial^2 G/\partial \xi^2$.

Variant 1. Let $G_{\xi\xi} \neq 0$; then Eq. (2.6) with (2.5) can be rewritten as the quasilinear equation

$$a_1 g_{xx} + a_2 g_{xz} + a_3 = 0 \tag{2.7}$$

with coefficients $a_i = b_i G_{\xi\xi\xi} - c_i G_{\xi\xi}$ ($i = 1, 2, 3$). The last equation, together with Eq. (2.5), can be considered as a system of linear equations in x and y with a determinant equal to $G_{\xi\xi}\alpha\widehat{\alpha}$. If $\widehat{\alpha} \neq 0$, the two equations can be solved for x and y:

$$x = \Phi_1(g_{xx}, g_{xz}, g_x, z, t), \quad y = \Phi_2(g_{xx}, g_{xz}, g_x, z, t). \tag{2.8}$$

Differentiating these equations with respect to x and y, substituting the expressions for g_y, g_{xy}, g_{xyz}, and g_{xxy} into them, and eliminating g_{xxx} from two of them, we arrive at the relations

$$\Phi_{1,1}G_{\xi\xi}g_{xz}^2 + \Phi_{1,2}(2\alpha_z + G_{\xi\xi}g_{xx}g_{xz} + G_z g_{xx}) = -G_\xi,$$

$$\Phi_{2,1}G_{\xi\xi}g_{xz}^2 + \Phi_{2,2}(2\alpha_z + G_{\xi\xi}g_{xx}g_{xz} + G_z g_{xx}) = 1,$$

where $\Phi_{i,1} = \partial\Phi_i/\partial g_{xx}$ and $\Phi_{i,2} = \partial\Phi_i/\partial g_{xz}$. Using the expressions for the functions Φ_i ($i = 1, 2$) from the last equations, we find $g_{xx} = \Psi_1(g_x, z, t)$ and $g_{xz} = \Psi_2(g_x, z, t)$. Introducing these derivatives into (2.8), we obtain the contradictory equalities $x = \widehat{\Phi}_1(g_x, z, t)$ and $y = \widehat{\Phi}_2(g_x, z, t)$; therefore, in this case, one should consider $\widehat{\alpha} = 0$ or

$$\frac{\partial \alpha}{\partial t} + 2f\frac{\partial \alpha}{\partial z} - \frac{\partial^2 \alpha}{\partial z^2} - 2\alpha\frac{\partial f}{\partial z} = 0.$$

Then, the coefficients a_i, b_i, c_i ($i = 1, 2, 3$), b_4, and b_5 do not depend on x and y explicitly. A further analysis of equations (2.5) and (2.7) reveals that they can be solved for the second derivatives g_{xx} and g_{xz}:

$$g_{xx} = \Phi_1(g_x, z, t), \qquad g_{xz} = \Phi_2(g_x, z, t). \tag{2.9}$$

Variant 1.1. Let $\Phi_1 \neq 0$ in Eqs. (2.9); integrating the first equation in (2.9) over x, we have $\widehat{\Phi}(g_x, z, t) = x + q(y, z, t)$ or

$$g_x = \Phi(x + q(y, z, t), z, t) \tag{2.10}$$

where $q(y, z, t)$ is an arbitrary function of integration, and $\Phi(\lambda, z, t)$ is a certain function. The function $g(x, y, z, t)$ must also satisfy Eq. (2.4) and the second equation in (2.9), i.e., $g_y = 2x\alpha + \Phi_3(x + q(y, z, t), z, t)$ and $g_{xz} = \widehat{\Phi}_2(x + q(y, z, t), z, t)$. An investigation of the compatibility of the last equations and (2.10) leads to the representation

$$g = -y^2\alpha k(t) + y\lambda(z, t) + \mu(x + yk(t), z, t). \tag{2.11}$$

Since, in all the representations for the starting variables of pressure and velocity, it is necessary to know only the derivatives of the function μ with respect x and y, we introduce the notation $\varphi(\eta, z, t) = (\partial\mu/\partial\eta)(\eta, z, t)$. Substituting (2.11) into Eqs. (2.2) and separating the powers of y, we find that $k' = 0$, i.e., $k = \text{const}$, while the functions $\alpha(z,t)$, $\lambda(z,t)$, and $\varphi(\eta, z, t)$ satisfy the equations

$$\frac{\partial\alpha}{\partial t} + 2f\frac{\partial\alpha}{\partial z} - \frac{\partial^2\alpha}{\partial z^2} = 2\alpha\frac{\partial f}{\partial z}, \qquad \frac{\partial^2 f}{\partial z\partial t} + 2f\frac{\partial^2 f}{\partial z^2} - \frac{\partial^3 f}{\partial z^3} = \alpha^2 + \left(\frac{\partial f}{\partial z}\right)^2,$$

$$\frac{\partial\lambda}{\partial t} + 2f\frac{\partial\lambda}{\partial z} - \frac{\partial^2\lambda}{\partial z^2} = \lambda(\alpha + f_z), \tag{2.12}$$

$$\frac{\partial\varphi}{\partial t} + 2f\frac{\partial\varphi}{\partial z} - \frac{\partial^2\varphi}{\partial z^2} - (k^2+1)\frac{\partial^2\varphi}{\partial\eta^2} + (\lambda - \eta(\alpha + f_z))\frac{\partial\varphi}{\partial\eta} + (\alpha - f_z)\varphi = 0.$$

Here $\eta = x + ky$; moreover, $\psi = y(\lambda - \alpha\eta) + \mu$.

Variant 1.2. Let $\Phi_1 = 0$; then

$$g = xq_1(y,z,t) + q_2(y,z,t). \tag{2.13}$$

By virtue of Eq. (2.4), the functions $q_1(y,z,t)$ and $q_2(y,z,t)$ become

$$g = 2\alpha xy + x\widehat{\lambda}(z,t) + \widehat{\mu}(2\alpha y + \widehat{\lambda}(z,t), z, t).$$

We note that, if $\alpha = 0$, this representation is a particular case of the representation (2.11); therefore, we assume here that $\alpha \neq 0$. Then the equation $g_{xz} = \Phi_2(g_x, z, t)$ leads to a further concrete definition of the representation (2.13):

$$g = 2\alpha x(y + \lambda(z,t)) + \mu(y,z,t).$$

As above, $\varphi = \partial\mu/\partial y$. Substituting this form of solution into the starting equations (2.2) and separating them with respect to variable x, we obtain the following equations for functions $\alpha(z,t)$, $\lambda(z,t)$, and $\varphi(y,z,t)$:

$$\frac{\partial\alpha}{\partial t} + 2f\frac{\partial\alpha}{\partial z} - \frac{\partial^2\alpha}{\partial z^2} = 2\alpha\frac{\partial f}{\partial z}, \qquad \frac{\partial^2 f}{\partial z\partial t} + 2f\frac{\partial^2 f}{\partial z^2} - \frac{\partial^3 f}{\partial z^3} = \alpha^2 + \left(\frac{\partial f}{\partial z}\right)^2,$$

$$\frac{\partial\lambda}{\partial t} + 2f\frac{\partial\lambda}{\partial z} - \frac{\partial^2\lambda}{\partial z^2} - \frac{2}{\alpha}\frac{\partial\alpha}{\partial z}\frac{\partial\lambda}{\partial z} = -\lambda\left(\frac{\partial f}{\partial z} - \alpha\right),$$

$$\frac{\partial^2\varphi}{\partial y\partial t} + 2f\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^3\varphi}{\partial y\partial z^2} - \frac{\partial^3\varphi}{\partial y^3} - (y(f_z + \alpha) + 2\alpha\lambda)\frac{\partial^2\varphi}{\partial y^2} + (\alpha - f_z)\frac{\partial\varphi}{\partial y} = 0.$$

Here $\psi = \alpha xy + 2\alpha x\lambda + \mu$.

Variant 2. Let $G_{\xi\xi} = 0$ or $G = a(z,t)g_x + \lambda(z,t)$. This means that the function $g(x,y,z,t)$ must satisfy the equation

$$g_y - a(z,t)g_x = 2\alpha x + \lambda(z,t). \tag{2.14}$$

The solution of the last equation depends on the value of the function $a(z,t)$ and Eqs. (2.5) and (2.6) take the form

$$\begin{aligned} &g_{xx}a_z + \alpha(-af_z + a\alpha - b) + 2\alpha_z a_z = 0, \\ &2g_{xz}a_z - \widehat{\alpha}(x + ya) - g_x b - \lambda f_z - (\lambda_t + 2f\lambda_z - \lambda_{zz}) - \alpha\lambda = 0, \end{aligned} \tag{2.15}$$

where $b \equiv a_t + 2fa_z - a_{zz}$

Variant 2.1. If $a = 0$, the general solution of Eq. (2.14) can be represented in the form

$$g = 2\alpha xy + y\lambda + \mu(x,z,t).$$

Substitution of this representation into (2.2) yields

$$\frac{\partial\alpha}{\partial t} + 2f\frac{\partial\alpha}{\partial z} - \frac{\partial^2\alpha}{\partial z^2} = 2\alpha\frac{\partial f}{\partial z}, \qquad \frac{\partial^2 f}{\partial z\partial t} + 2f\frac{\partial^2 f}{\partial z^2} - \frac{\partial^3 f}{\partial z^3} = \alpha^2 + \left(\frac{\partial f}{\partial z}\right)^2,$$

$$\frac{\partial \lambda}{\partial t} + 2f\frac{\partial \lambda}{\partial z} - \frac{\partial^2 \lambda}{\partial z^2} - \lambda\Big(\frac{\partial f}{\partial z} - \alpha\Big) = 0, \tag{2.16}$$

$$\frac{\partial \varphi}{\partial t} + 2f\frac{\partial \varphi}{\partial z} - \frac{\partial^2 \varphi}{\partial z^2} - \frac{\partial^2 \varphi}{\partial x^2} + (\lambda + x(\alpha - f_z))\frac{\partial \varphi}{\partial x} - \Big(\alpha + \frac{\partial f}{\partial z}\Big)\varphi = 0,$$

where $\varphi = \partial\mu/\partial x$.

Variant 2.2. If $a \neq 0$, the solution of Eq. (2.14) has the form

$$g = \frac{\alpha x^2 + x\lambda}{a} + \mu(x + ya, z, t),$$

in which $\mu(x + ya, z, t)$ is so far an arbitrary function.

If $a_z \neq 0$, the first equation in (2.15) implies that the function $g(x, y, z, t)$ is a quadratic polynomial in x and y. This corresponds to the linear velocity profile relative to x and y [7]. If $a_z = 0$, but $a_t \neq 0$, again it follows from the second equation in (2.15) that the function $g(x, y, z, t)$ is a quadratic polynomial in x and y. Therefore, it is necessary to consider the case where the function $a(z,t)$ is constant, the functions $\alpha(z,t)$, $\lambda(z,t)$, and $f(z,t)$ satisfy the same equations (2.16) as in the previous case, and the function $\varphi(\eta, z, t) = (\partial\mu/\partial\eta)(\eta, z, t)$ must satisfy the equation

$$\frac{\partial \varphi}{\partial t} + 2f\frac{\partial \varphi}{\partial z} - \frac{\partial^2 \varphi}{\partial z^2} - (a^2 + 1)\frac{\partial^2 \varphi}{\partial \eta^2} + (\lambda + \eta(\alpha - f_z))\frac{\partial \varphi}{\partial \eta} - (\alpha + f_z)\varphi = 0.$$

In all the above systems which correspond to the hyperbolic Monge–Ampere equation, the following two equations are split off:

$$\frac{\partial \alpha}{\partial t} + 2f\frac{\partial \alpha}{\partial z} - \frac{\partial^2 \alpha}{\partial z^2} = 2\alpha\frac{\partial f}{\partial z}, \qquad \frac{\partial^2 f}{\partial z \partial t} + 2f\frac{\partial^2 f}{\partial z^2} - \frac{\partial^3 f}{\partial z^3} = \alpha^2 + \Big(\frac{\partial f}{\partial z}\Big)^2, \tag{2.17}$$

which can be investigated independently of the other equations. The infinite group G^∞ admitted by system (2.17) corresponds to the algebra L^∞, which consists of the operators

$$Z_h = 2h(t)\partial_z + h'(t)\partial_f, \qquad T = \partial_t, \qquad R = 2t\partial_t + z\partial_z - 2\alpha\partial_\alpha - f\partial_f$$

with an arbitrary function $h(t) \in C^\infty$. The commutators of these operators are of the form $[T, Z_h] = Z_{h'}$, $[R, Z_h] = Z_{2th'-h}$, $[T, R] = 2T$, and $[Z_{h_1}, Z_{h_2}] = 0$ (here and henceforth, the prime denotes differentiation of the function of one variable with respect to its argument).

Since to construct the invariant and the partially invariant solutions of system (2.17), knowledge of invariants of the corresponding groups is necessary, we classify the subalgebras by the availability of invariants of definite forms.

All the subalgebras of the infinite-dimensional algebra L^∞ can be of two types: those which contain the extension operator R and those which do not contain one, i.e., $(R + cT + Z_{h_1}) \oplus L$ or L. Here L is the subalgebra of the infinite-dimensional algebra consisting of the operators T and Z_h. We note that the constant c in the operator $R + cT + Z_{h_1}$ can be set to zero. This corresponds to the replacement of the time t by the time $t + c$ in the invariants.

In turn, subalgebras L can also be of two types: those which involve the operator T and those which do not involve one, i.e., $(T + Z_{h_1}) \oplus \bar{L}$ or $\bar{L}$. Here $\bar{L}$ is the subalgebra of the infinite-dimensional algebra consisting of operators Z_h.

First of all, we consider the possible invariants of the subalgebras L. If the dimensionality of the first-class subalgebra $(T + Z_{h_1}) \oplus \bar{L}$ is greater than 2, they have only one invariant α, while the second-class subalgebras $\bar{L}$ of a dimensionality greater than 1 have two invariants: α and t. The invariants for the other subalgebras $\{T + Z_{h_1}, Z_{h_2}\}$, $\{T + Z_{h_1}\}$, and $\{Z_h\}$ can easily be found. The invariants for the subalgebra $\{T + Z_{h_1}, Z_{h_2}\}$ are α and $f - cz - H(t)$, where $H(t) = h_1(t) - 2c\int h_1(t)\,dt$. Here c is a constant following from the requirement that the above operators form a subalgebra. This requirement corresponds to the condition $[T + Z_{h_1}, Z_{h_2}] = 2cZ_{h_2'}$ and is expressed by the equality $h_2' = 2ch_2$, which imposes a constraint on the function

h_2. For the subalgebra $\{T + Z_{h_1}\}$, the invariants are $z - 2H(t), \alpha$, and $f - H'(t)$, where $H(t)$ is the integral of the function h_1. The subalgebra $\{Z_h\}$ has the invariants t, α, and $f - zH(t)$, where $H(t) = h'/(2h)$.

We now consider all possible invariants of the subalgebras $(R + Z_h) \oplus L$. If a subalgebra L is the subalgebra of the first class $(T + Z_{h_1}) \oplus \bar{L}$ and is of dimensionality greater than 2, this subalgebra $(R + Z_h) \oplus L$ has no invariants. If L is the subalgebra of the second class $\bar{L}$ of dimensionality greater than 1, the subalgebra $(R + Z_{h_1}) \oplus L$ has the only invariant $t\alpha$. The invariants of the remaining subalgebras $\{T + Z_{h_1}, Z_{h_2}, R + Z_{h_3}\}$, $\{T + Z_{h_1}, R + Z_{h_2}\}$, $\{Z_{h_1}, R + Z_{h_2}\}$, $\{R + Z_h\}$, and $\{R + Z_h\}$ can easily be found as well.

For subalgebras of the form $\{T + Z_{h_1}, Z_{h_2}, R + Z_{h_3}\}$, the subalgebra condition constrains the functions $h_1(t), h_2(t)$, and $h_3(t)$:

$$h_1(t) = H(t) - c, \quad h_2 = 1, \quad h_3'(t) = 2tH'(t) + H(t)$$

with a certain constant c and function $H(t)$. These subalgebras have the only invariant $\alpha(f - H(t))^{-2}$.

For subalgebras of the form $\{T + Z_{h_1}, R + Z_{h_2}\}$, the subalgebra condition requires that the functions $h_1(t)$ and $h_2(t)$ satisfy the relations $h_1(t) = H'(t)$ and $h_2(t) = 2tH'(t) - H(t)$, where $H(t)$ is a certain function. The invariants of these subalgebras are $\alpha(z - 2H(t))^2$ and $(f - H'(t))(z - 2H(t))$.

For subalgebras of the form $\{Z_{h_1}, R + Z_{h_2}\}$, the subalgebra condition is $th_1'(t) = 2ch_1$, where c is a certain constant, while the invariants are $t\alpha$ and $(f - H(t) - cz/t)t^{1/2}$. Here

$$H(t) \doteq \frac{1}{2t}h_2(t) - \frac{(4c-1)t^{-1/2}}{2} \int t^{-3/2} h_2(t)\, dt.$$

Finally, the subalgebra $\{R + Z_h\}$ has the invariants $(z - 2H(t))t^{-1/2}$, $t\alpha$, and $(f - H'(t))t^{1/2}$, where $h(t) = 2tH'(t) - H(t)$.

Thus, all the invariants of the subalgebras L^∞ can be of ten types indicated above: 1) α; 2) $t\alpha$; 3) $\alpha(f - H(t))^{-2}$; 4) t and α; 5) α and $f - cz - H(t)$; 6) $\alpha(z - 2H(t))^2$ and $(f - H'(t))(z - 2H(t))$; 7) $t\alpha$ and $(f - H(t) - cz/t)t^{1/2}$; 8) $z - 2H(t)$, α, and $f - H'(t)$; 9) t, α and $f - zH(t)$; 10) $(z - 2H(t))t^{-1/2}$, $t\alpha$, and $(f - H'(t))t^{1/2}$ with a certain function $H(t)$ and constant c.

3. Invariant Solutions of System (2.17). Analysis of the compatibility conditions for partially invariant solutions which correspond to invariants 1–4 shows that these solutions reduce either to the case where $\alpha = 0$, or to one of the invariant solutions obtained for invariants 5–10, namely: for type 1, $\alpha = 0$, and for types 2 and 4, $\alpha = 0$ or the solution is the invariant solution of type 9, and for type 3, a solution exists only if $\alpha = 0$.

Type 5. There is only one invariant solution $\alpha = 0$ and $f = H(t)$, where $H(t)$ is an arbitrary function of time.

Type 6. The invariant solution has the representation

$$\alpha = \frac{c_1}{(z - 2H(t))^2}, \qquad f = H'(t) + \frac{c_2}{z - 2H(t)},$$

where c_1 and c_2 are constants which satisfy the relations $c_1(c_2 + 3) = 0$ and $3c_2(c_2 + 2) = c_1^2$.

Type 7. The invariant solution has the representation $\alpha = c_1/t$ and $f = H(t) + c_2 z/t$, where c_1 and c_2 are constants which satisfy the relations $c_1(2c_2 + 1) = 0$ and $c_2(c_2 + 1) + c_1^2 = 0$.

Type 8. The invariant solution has the representation $\alpha = \alpha(\lambda)$, $f = H'(t) + \phi(\lambda)$, and $\lambda = z + 2q(t)$. Substitution into Eqs. (2.17) yields the S/H system of ordinary differential equations for the functions $\alpha(\lambda)$ and $\phi(\lambda)$

$$\alpha'' - 2\phi\alpha' + 2\alpha\phi' = 0, \qquad \phi''' - 2\phi\phi'' + (\phi')^2 + \alpha^2 = 0.$$

Type 9. The invariant solution has the representation $\alpha = \alpha(t)$ and $f = zH(t) + \phi(t)$. Substituting this into Eqs. (2.17), we obtain $\alpha' = 2\alpha H'$ and $H' = \alpha^2 + H^2$. The new solution of the last equations, which differs from type 7, is of the form (up to the time shift t)

$$\alpha = \frac{c}{c^2t^2 - 1}, \qquad H = -\frac{c^2 t}{c^2t^2 - 1},$$

where c is a constant.

Type 10. The invariant solution has the representation $\alpha = t^{-1}P(\lambda)$, $f = H'(t) + t^{-1/2}Q(\lambda)$, and $\lambda = t^{-1/2}(z - 2H(t))$. Substitution into Eqs. (2.17) yields the following S/H system of ordinary differential equations for the functions $P(\lambda)$ and $Q(\lambda)$:

$$2P'' + (\lambda - 4Q)P' + 4PQ' + 2P = 0, \quad 2Q''' + (\lambda - 4Q)Q'' + 2(Q')^2 + 2Q' + 2P^2 = 0.$$

4. The Group Stratification of System (2.17). The infinite-dimensional group which corresponds to the operator Z_h has the enhanced operator $2h\partial_z + h'(t)(\partial_f - 2f_z\partial_{f_t} - 2\alpha_z\partial_{\alpha_t} - 2\beta_z\partial_{\beta_t}) + h''\partial_{f_t}$, where $\beta = f_z$. The universal invariant of the first order (all calculations are similar to those performed during group stratification of the system of equations of the plane steady boundary layer [16]) is easily obtained:

$$\underset{1}{J} = (t, \beta, \alpha, \alpha_z, \beta_z, \beta_t + 2f\beta_z,\ \alpha_t + 2f\alpha_z);$$

therefore, the AG system of rank 2 can be written as follows:

$$\alpha = \alpha(t,\beta), \quad \alpha_z = \varphi(t,\beta), \quad \beta_z = \gamma(t,\beta), \quad \beta_t + 2f\beta_z = \xi(t,\beta), \quad \alpha_t + 2f\alpha_z = \zeta(t,\beta), \tag{4.1}$$

where $\alpha(t,\beta)$, $\varphi(t,\beta)$, $\gamma(t,\beta)$, $\xi(t,\beta)$ and $\zeta(t,\beta)$ are the unknown functions. The compatibility conditions for the last system and the starting system (2.17) have the form

$$\varphi = \gamma\alpha_\beta, \quad \zeta = \gamma\varphi_\beta + 2\alpha\beta, \quad \xi = \gamma\gamma_\beta + \alpha^2 + \beta^2; \tag{4.2}$$

$$\alpha_t + (\alpha^2 + \beta^2)\alpha_\beta - \gamma^2\alpha_{\beta\beta} = 2\alpha\beta, \qquad \gamma_t + (\alpha^2 + \beta^2)\gamma_\beta - \gamma^2\gamma_{\beta\beta} = 2\alpha\gamma\alpha_\beta. \tag{4.3}$$

Thus, the group stratification of system (2.17) relative to the infinite-dimensional group with operator Z_h is the union of the automorphic system (4.1) with functions (4.2) and the governing system which consists of Eqs. (4.3).

Remark 1. System (4.1), (4.3) is equivalent to the starting system (2.17) provided that $f_{zz} \neq 0$. The case where $f_{zz} = 0$ corresponds to the above considered invariant solution $\alpha = \alpha(t)$ and $f = zq(t) + \phi(t)$.

System (4.3) admits only a two-parameter group which corresponds to the operators $T = \partial_t$ and $\bar{R} = t\partial_t - \beta\partial_\beta - \alpha\partial_\alpha - (3/2)\psi\partial_\psi$. The optimal system of subalgebras of the algebra $L_2 = \{T, \bar{R}\}$ consists of only three representatives: L_2, $\{T\}$, and $\{\bar{R}\}$. The invariant solutions of system (4.3) relative to the algebra L_2 lead to solutions which correspond to the invariant solutions of type 7 for system (2.17). The invariant solutions of system (4.3) relative to the subalgebra $\{T\}$ lead to solutions which correspond to the invariant solutions of type 9 for system (2.17). Finally, the invariant solutions of system (4.3) relative to the subalgebra $\{\bar{R}\}$ can be represented as $\alpha = t^{-1}A(\lambda)$, and $\psi = t^{-3/2}B(\lambda)$, where $\lambda = t\beta$ and the functions $A(\lambda)$ and $B(\lambda)$ satisfy the S/H system:

$$B^2A'' - (A^2 + \lambda^2 + \lambda)A' + (1 + 2\lambda)A = 0,$$

$$B^2B'' - (A^2 + \lambda^2 + \lambda)B' + \Big(\frac{3}{2} + AA'\Big)B = 0.$$

Remark 2. Apparently, the last type of invariant solution of system (4.3) corresponds to the invariant solutions of system (2.17) of type 10; however, the search for this invariant solution is preferred, since after the solution of the governing system is found, it is necessary to construct the solution of the automorphic system.

5. Certain Solutions of System (2.12). We give here some particular solutions of system (2.12) and, consequently, some solutions of the three-dimensional Navier–Stokes equations of the above class of partially invariant solutions.

As an example, we consider a solution of the form $\varphi = s(t,z)g(\eta)$ (the case where $g \neq$ const is of interest). Substitution of this form of solution into the last equation (2.12) yields

$$-(k^2 + 1)g'' + (\lambda - \eta(\alpha + f_z))g' + ag = 0, \tag{5.1}$$

where $a = s^{-1}(s_t + 2fs_z - s_{zz} + (\alpha - f_z)s)$. After differentiation of Eq. (5.1) with respect to t and z we obtain the consequences

$$(\lambda_t - \eta(\alpha + f_z)_t)g' + a_t g = 0, \qquad (\lambda_z - \eta(\alpha + f_z)_z)g' + a_z g = 0. \tag{5.2}$$

Inasmuch as $g \neq 0$, the determinant of the last system, which is considered as a system of linear equations for g and g', vanishes, which gives the conditions $\lambda_t a_z - \lambda_z a_t = 0$ and $(\alpha + f_z)_t a_z - (\alpha + f_z)_z a_t = 0$.

For $a = \text{const}$, from Eqs. (5.2) we find that $\lambda = \text{const}$ and $\alpha + f_z = \text{const}$. However, the first and second equations of system (2.12) yield $\alpha = -f_z$. Equation (5.1) is then the ordinary second-order differential equation for $g(\eta)$ with constant coefficients

$$-(k^2 + 1)g'' + \lambda g' + ag = 0. \tag{5.3}$$

Thereby, the prescribed-form solution of system (2.12) is determined, namely, $\alpha = -f_z$, λ is an arbitrary constant, the function $g(\eta)$ is found by solving Eq. (5.3), and the functions $f(t, z)$ and $s(t, z)$ must satisfy the equations $f_{tz} + 2ff_{zz} - f_{zzz} = 0$ and $s_t + 2fs_z - s_{zz} - (2f_z + a)s = 0$, where a is an arbitrary constant.

The case $a \neq \text{const}$ is possible only for $g = \exp(-c\eta)$, $\alpha = -f_z$, and $\lambda = ac^{-1} - c(k^2 + 1)$ with the arbitrary constant $c \neq 0$. Moreover, the functions $a(t, z)$, $f(t, z)$, and $s(t, z)$ must satisfy the recursive system of differential equations

$$f_{tz} + 2ff_{zz} - f_{zzz} = 0, \qquad a_t + 2fa_z - a_{zz} = 0, \qquad s_t + 2fs_z - s_{zz} - (2f_z + a)s = 0.$$

Remark 3. Other types of solution of system (2.12) with separable variables are also possible. For example, an analysis of solutions of the form $\varphi = s(t, \eta) \neq 0$ shows that they are admitted only in the case where the solution of the first two equations of system (2.17) is the invariant solution of types 7 or 9.

Remark 4. The last equation of system (2.12) is linear; moreover, it is the only equation that involves three independent variables. One can easily see that the solutions $\varphi(\eta, z, t)$ of this equation, in which the functions f, α, and λ of the variables z and t are assumed to be known, involve polynomials of any degree in variable η. The coefficients $\varphi_j(z, t)$ of these polynomials satisfy the recursive system of parabolic equations, which is not given here.

6. Interpretation of the Partially Invariant Solutions Considered Above. We show that the solutions described in Sec. 2 can be interpreted as the motions with the plane free boundary $z = l(t)$, where the function l, together with the functions u, v, w, and p, should be determined.

First of all, we note that, for the known $f(z, t)$, the function $p(z, t)$ is found by means of quadratures up to the additive function $\chi(t)$. By virtue of the arbitrariness of χ, the dynamic condition for the normal stress at the free boundary $-p + 2\partial w/\partial z = 0$ for $z = l(t)$ can always be satisfied after the functions $w = 2f$ and l have been found.

The kinematic condition at the free boundary has the form $dl/dt = 2f[l(t), t]$ and the conditions of the absence of tangential stresses, which are initially formulated in terms of functions u, v, and w, lead to the equalities

$$\psi_{yz} - xf_{zz} = 0, \qquad \psi_{xz} + yf_{zz} = 0 \quad \text{for} \quad z = l(t). \tag{6.1}$$

Separation of the last equations with respect to the variables x and y forms the boundary conditions for the functions which enter system (2.12). For example, in Variant 1.1 (see Sec. 2), relations (6.1) are identically satisfied in x and y if the conditions $\alpha_z = f_{zz} = \lambda_z = \varphi_z = 0$ hold for $z = l(t)$. Let the second surface, which bounds the fluid, be the plane $z = 0$. If the equalities

$$\alpha = f = f_z = \lambda = \varphi = 0 \quad \text{for} \quad z = 0 \tag{6.2}$$

hold, the adhesion conditions are satisfied on this plane, and it can be identified with an immovable solid boundary.

Other interpretations are also possible for the solutions corresponding to Variant 1.1 (Sec. 2), which is considered here for definiteness. In particular, if conditions (6.2) for the functions λ and φ are replaced by the inhomogeneous conditions $\lambda + k\varphi = \sigma(t)$ and $-\varphi = \tau(t)$ for $z = 0$ (σ and τ are arbitrary functions of t), the adhesion conditions are satisfied on a solid plane $z = 0$, which moves translationally with a speed $\boldsymbol{V} = (\sigma, \tau)$.

Moreover, the second plane, which bounds the liquid layer, can also be a moving free boundary $z = m(t)$.

Remark 5. System (2.12) admits particular solutions in which $\alpha = f_z$ and $\lambda = \varphi = 0$. They relate to the planar motions of a fluid which describe, in particular, the process of symmetrical deformation of a viscous strip $|z| < l(t)$ with free boundaries. This process is dealt with in the papers [5, 10, 11].

Remark 6. We have considered only the hyperbolic case of the Monge–Ampere equation (2.3). At the same time, this equation is compatible with system (2.2) in the elliptic case as well if ψ is assumed to be a quadratic polynomial in the variables x and y. Among the partially invariant solutions of system (1.1) obtained in this way, there are solutions which describe the spreading of a viscous fluid with free boundary $z = l(t)$ on the plane $z = 0$ which rotates about the z axis with a specified angular velocity $\Omega(t)$. This problem was studied in [5, 8, 9].

REFERENCES

1. V. O. Bytev, "Group properties of the Navier–Stokes equations," in: *Numerical Methods in Mechanics of Continuous Media* (collected scientific papers) [in Russian], Novosibirsk, bf 3, No. 3 (1972), pp. 13–17.
2. V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachev, and A. A. Radionov, *Application of Group-Theoretic Methods in Hydrodynamics* [in Russian], Nauka, Novosibirsk (1994).
3. L. V. Ovsyannikov, *Group Analysis of Differential Equations* [in Russian], Nauka, Moscow (1994).
4. T. von Kármán, "Uber laminare und turbulente reibung," *ZAMM*, bf 1, No. 4, 233–252 (1921).
5. V. V. Pukhnachev, "Nonstationary motions of a viscous fluid with a free boundary described by partially invariant solutions of the Navier–Stokes equations," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], Novosibirsk, **10** (1972), pp. 125–137.
6. L. V. Ovsyannikov, "Partial invariance," *Dokl. Akad. Nauk SSSR*, **186**, No. 1, 22—25 (1969).
7. A. F. Sidorov, "Two classes of solutions of fluid and gas mechanics equations and their connection to the theory of traveling waves," *Prikl. Mekh. Tekh. Fiz.*, No. 2, 34–40 (1989).
8. O. M. Lavrent'eva, "Flow of a viscous fluid in a layer on a rotating plane," *Prikl. Mekh. Tekh. Fiz.*, No. 5, 41–48 (1989).
9. O. M. Lavrent'eva and G. B. Volkova, "Limiting regimes of spreading of a layer on a rotating plane," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], Novosibirsk, **111** (1994), pp. 68–77.
10. V. V. Pukhnachov, "Nonstationary viscous flows with a cylindrical free surface," in: *Progress in Nonlinear Differential Equations and Their Applications*, V. 35: *Topics in Nonlinear Analysis: The Herbert Amann Anniversary Volume*, Birkhäuser Verlag, Basel (1998), pp. 553–569.
11. V. A. Galaktionov and J. L.Vazquez, "Blow-up of a class of solutions with free boundary for the Navier–Stokes equations," *Advan. Differ. Equations* (in press).
12. L. V. Ovsyannikov, "Regular and nonregular, partially invariant solutions," *Dokl. Akad. Nauk SSSR*, **343**, No. 2, 156—159 (1995).
13. L. V. Ovsyannikov and A. P. Chupakhin, "Regular, partially invariant submodels of the gas dynamics equations," *Prikl. Mat. Mekh.*, **60**, No. 6, 990—999 (1996).
14. E. Goursat, *Cours d'Analyse Mathematique*, Paris (1927).
15. A. C. Hearn, *REDUCE Users Manual*, Ver. 3.3, CP 78, Rand Corporation, Santa Monica (1987).
16. L. V. Ovsyannikov, "Group Stratification of boundary-layer equations," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], Novosibirsk, **1** (1969), pp. 24–35.